On the Theory of Long Waves and Bores.

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In the theory of long waves in two dimensions, which we may suppose to be reduced to a "steady" motion, it is assumed that the length is so great in proportion to the depth of the water that the velocity in a vertical direction can be neglected, and that the horizontal velocity is uniform across each section of the canal. This, it should be observed, is perfectly distinct from any supposition as to the height of the wave. If l be the undisturbed depth, and h the elevation of the water at any point of the wave, u_0, u the velocities corresponding to l, $l+h$ respectively, we have, as the equation of continuity,

$$u = \frac{lu_0}{l+h}. \tag{1}$$

By the principles of hydrodynamics, the increase of pressure due to retardation will be

$$\tfrac{1}{2}\rho(u_0^2-u^2) = \frac{\rho u_0^2}{2} \cdot \frac{2lh+h^2}{(l+h)^2}. \tag{2}$$

On the other hand, the loss of pressure (at the surface) due to height will be $g\rho h$, and therefore the total gain of pressure over the undisturbed parts is

$$\left(\frac{\rho u_0^2}{l} \cdot \frac{1+h/2l}{(1+h/l)^2} - g\rho\right) h. \tag{3}$$

If, now, the ratio h/l be very small, the coefficient of h becomes

$$\rho\,(u_0^2/l-g), \tag{4}$$

and we conclude that the condition of a free surface is satisfied, provided $u_0^2 = gl$. This determines the rate of flow u_0, in order that a stationary wave may be possible, and gives, of course, at the same time the velocity of a wave in still water.

Unless h^2 can be neglected, it is impossible to satisfy the condition of a free surface for a stationary long wave—which is the same as saying that it is impossible for a long wave of finite height to be propagated in still water without change of type.

Although a constant gravity is not adequate to compensate the changes of pressure due to acceleration and retardation in a long wave of finite height, it is evident that complete compensation is attainable if gravity be made a suitable function of height; and it is worth while to enquire what the law of force must be in order that long waves of unlimited height may

travel with type unchanged. If f be the force at height h, the condition of constant surface pressure is

$$\tfrac{1}{2}\rho u_0^2\left\{1-\frac{l^2}{(l+h)^2}\right\} = \rho\int_0^h f\,dh\,; \tag{5}$$

whence

$$f = -\frac{u_0^2}{2}\cdot\frac{d}{dh}\frac{l^2}{(l+h)^2} = u_0^2\frac{l^2}{(l+h)^3}, \tag{6}$$

which shows that the force must vary inversely as the cube of the distance from the bottom of the canal. Under this law the waves may be of any height, and they will be propagated unchanged with the velocity $\sqrt{(f_1 l)}$, where f_1 is the force at the undisturbed level.*

It may be remarked that we are concerned only with the values of f at water-levels which actually occur. A change in f below the lowest water-level would have no effect upon the motion, and thus no difficulty arises from the law of inverse cube making the force infinite at the bottom of the canal.

When a wave is limited in length, we may speak of its velocity relatively to the undisturbed water lying beyond it on the two sides, and it is implied that the uniform levels on the two sides are the same. But the theory of long waves is not thus limited, and we may apply it to the case where the uniform levels on the two sides of the variable region are different, as, for example, to *bores.* This is a problem which I considered briefly on a former occasion,† when it appeared that the condition of conservation of energy could not be satisfied with a constant gravity. But in the calculation of the loss of energy a term was omitted, rendering the result erroneous, although the general conclusions are not affected. The error became apparent in applying the method to the case above considered of a gravity varying as the inverse cube of the depth. But, before proceeding to the calculation of energy, it may be well to give the generalised form of the relation between velocity and height which must be satisfied in a *progressive* wave,‡ whether or not the type be permanent.

In a small positive progressive wave, the relation between the particle-velocity u at any point (now reckoned relatively to the parts outside the wave) and the elevation h is

$$u = \sqrt{(f/l)}\,.\,h. \tag{7}$$

If this relation be violated anywhere, a wave will emerge, travelling in the

* 'Phil. Mag.,' vol. 1, p. 257 (1876); 'Scientific Papers,' vol. 1, p. 254.

† 'Roy. Soc. Proc.,' A, vol. 81, p. 448 (1908); 'Scientific Papers,' vol. 5, p. 495.

‡ Compare 'Scientific Papers,' vol. 1, p. 253 (1899).

negative direction. In applying (7) to a wave of finite height, the appropriate form of (7) is

$$du = \sqrt{\left(\frac{f}{l+h}\right)}\, dh, \tag{8}$$

where f is a known function of $l+h$, or on integration

$$u = \int_0^h \sqrt{\left(\frac{f}{l+h}\right)}\, dh. \tag{9}$$

To this particle-velocity is to be added the wave-velocity

$$\sqrt{\{(l+h)f\}}, \tag{10}$$

making altogether for the velocity of, *e.g.*, the crest of a wave relative to still water

$$\int_0^h \sqrt{\left(\frac{f}{l+h}\right)}\, dh + \sqrt{\{(l+h)f\}}. \tag{11}$$

Thus if f be constant, say g, (9) gives de Morgan's formula

$$u = 2\sqrt{g}\,\{(l+h)^{\frac{1}{2}} - l^{\frac{1}{2}}\}, \tag{12}$$

and from (11)

$$3\sqrt{g}\sqrt{(l+h)} - 2\sqrt{(gl)}. \tag{13}$$

If, again,

$$f = \frac{f_1 l^3}{(l+h)^3}, \tag{14}$$

(11) gives as the velocity of a crest

$$\frac{f_1^{\frac{1}{2}} l^{\frac{1}{2}} h}{l+h} + \frac{f_1^{\frac{1}{2}} l^{\frac{3}{2}}}{l+h} = \sqrt{(f_1 l)}, \tag{15}$$

which is independent of h, thus confirming what was found before for this law of force.

As regards the question of a bore, we consider it as the transition from a uniform velocity u and depth l to a uniform velocity u' and depth l', l' being greater than l. The first relation between these four quantities is that given by continuity, viz.,

$$lu = l'u'. \tag{16}$$

The second relation arises from a consideration of momentum. It may be convenient to take first the usual case of a constant gravity g. The mean pressures at the two sections are $\frac{1}{2}gl$, $\frac{1}{2}gl'$, and thus the equation of momentum is

$$lu\,(u-u') = \tfrac{1}{2}g\,(l'^2 - l^2). \tag{17}$$

By these equations u and u' are determined in terms of l, l':—

$$u^2 = \tfrac{1}{2}g\,(l+l')\,.\,l'/l, \qquad u'^2 = \tfrac{1}{2}g\,(l+l')\,.\,l/l'. \tag{18}$$

We have now to consider the question of energy. The difference of work

done by the pressures at the two ends (reckoned per unit of time and per unit of breadth) is $lu\,(\tfrac{1}{2}gl-\tfrac{1}{2}gl')$. And the difference between the *kinetic* energies entering and leaving the region is $lu\,(\tfrac{1}{2}u^2-\tfrac{1}{2}u'^2)$, the density being taken as unity. But this is not all. The *potential* energies of the liquid leaving and entering the region are different. The centre of gravity rises through a height $\tfrac{1}{2}(l'-l)$, and the gain of potential energy is therefore $lu\,.\,\tfrac{1}{2}g\,(l'-l)$. The whole *loss* of energy is accordingly

$$lu\,\{\tfrac{1}{2}gl-\tfrac{1}{2}gl'+\tfrac{1}{2}u^2-\tfrac{1}{2}u'^2-\tfrac{1}{2}g(l'-l)\} = lu\left\{gl-gl'+\tfrac{1}{4}g\,(l+l')\left(\frac{l'}{l}+\frac{l}{l'}\right)\right\}$$
$$= lu\,.\frac{g\,(l'-l)^3}{4ll'}. \tag{19}$$

This is much smaller than the value formerly given, but it remains of the same sign. "That there should be a loss of energy constitutes no difficulty, at least in the presence of viscosity; but the impossibility of a gain of energy shows that the motions here contemplated cannot be reversed."

We now suppose that the constant gravity is replaced by a force f, which is a function of y, the distance from the bottom. The pressures p, p' at the two sections are also functions of y, such that

$$p = \int_y^l f\,dy, \qquad p' = \int_y^{l'} f\,dy. \tag{20}$$

The equation of momentum replacing (17) is now

$$lu\,(u-u') = \int_0^{l'} p'\,dy-\int_0^l p\,dy = \Big[p'y\Big]_0^{l'}-\Big[py\Big]_0^l-\int_0^{l'} y\,\frac{dp'}{dy}\,dy+\int_0^l y\,\frac{dp}{dy}\,dy$$
$$= \int_0^{l'} yf\,dy-\int_0^l yf\,dy = \int_l^{l'} yf\,dy, \tag{21}$$

the integrated terms vanishing at the limits. This includes, of course, all special cases, such as $f =$ constant, or $f \propto y^{-3}$.

As regards the reckoning of energy, the first two terms on the left of (19) are replaced by

$$lu\left\{\frac{1}{l}\int_0^l p\,dy-\frac{1}{l'}\int_0^{l'} p'\,dy\right\}. \tag{22}$$

The third and fourth terms representing kinetic energy remain as before. For the potential energy we have to consider that a length u and depth l is converted into a length u' and depth l'. If we reckon from the bottom, the potential energy is in the first case

$$u\int_0^l dy\int_0^y f\,dy,$$

in which

$$\int_0^y f\,dy = \int_0^l f\,dy-\int_y^l f\,dy = p_0-p,$$

p_0 denoting the pressure at the bottom, so that the potential energy is

$$ul\left\{p_0-\frac{1}{l}\int_0^l p\,dy\right\}.$$

The difference of potential energies, corresponding to the fifth and sixth terms of (19), is thus

$$lu\left\{p_0-p_0'-\frac{1}{l}\int_0^l p\,dy+\frac{1}{l'}\int_0^{l'} p'\,dy\right\}. \tag{23}$$

The integrals in (23) compensate those of (22), and we have finally as the loss of energy

$$lu\{p_0-p_0'+\tfrac{1}{2}u^2-\tfrac{1}{2}u'^2\} = lu\left\{\tfrac{1}{2}u^2-\tfrac{1}{2}u'^2-\int_l^{l'} f\,dy\right\}. \tag{24}$$

It should be remarked that it is only for values of y between l and l' that f is effectively involved.

In the special case where $f=\mu y^{-3}$, equations (16), (21) give

$$u^2l^2=\mu, \qquad u'^2l'^2=\mu, \tag{25}$$

the introduction of which into (24) shows that, in this case, the loss of energy vanishes; all the conditions can be satisfied, even though there be no dissipation. The reversed motion is then equally admissible.

Experimental.

The formation of bores is illustrated by a very ordinary observation, probably not often thought of in this connection. Something of the kind may usually be seen whenever a stream of water from a tap strikes a horizontal surface. The experiment is best made by directing a vertically falling stream into a flat and shallow dish from which the water overflows.* The effective depth may be varied by holding a glass plate in a horizontal position under the water surface. Where the jet strikes, it expands into a thin sheet which diverges for a certain distance, and this distance diminishes as the natural depth of the water over the plate is made greater. The circular boundary where the transition from a small to a greater depth takes place constitutes a bore on a small scale. The flow may be made two-dimensional by limiting it with two battens held in contact with the glass. I have not attempted measures. On the smallest scale surface-tension doubtless plays a considerable part, but this may be minimised by increasing the stream, and correspondingly the depth of the water over the plate, so far as may be convenient.

* The tap that I employed gives a jet whose diameter is 6 mm. A much larger tap may need to be fitted with a special nozzle.—May 14.